

# Fission and Fusion in the New Localized Structures to the Integrable $(2 + 1)$ -Dimensional Higher-Order Broer–Kaup System

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By means of the Weiss–Tabor–Carnevale (WTC) truncation method and the general variable separation approach (GVSA), analytical investigation of the integrable  $(2 + 1)$ -dimensional higher-order Broer–Kaup (HBK) system shows, due to the possibility of selecting three arbitrary functions, the existence of interacting coherent excitations such as dromions, solitons, periodic solitons, etc. The interaction between some of the localized solutions are elastic because they pass through each other and preserve their shapes and velocities, the only change being the phase shift. However, as for some soliton models, completely non-elastic interactions have been found in this model. These non-elastic interactions are characterized by the fact that, at a specific time, one soliton may fission to two or more solitons; or on the contrary, two or more solitons will fuse to one soliton.

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**KEY WORDS:** Fission; fusion; localized structures;  $(2 + 1)$ -Dimensional Higher Order Broer-Kaup System; non-elastic interactions; general variable separation approach; variable separation approach.

## 1. INTRODUCTION

Much work has been done over the last year on the subject of obtaining special solutions of the nonlinear partial differential equations (NLPDEs), which govern many dynamic problems in physics and other fields. Due to its wide applications in mathematics, physics, chemistry, biology, communications, astrophysics and geophysics, the soliton theory plays a key role in this study. In the soliton theory, how to obtain these exact solutions of the NLPDEs is important. There are many

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powerful methods developed to solve the NLPDEs such as the inverse scattering transformation (Gardner *et al.*, 1976), the Hirota bilinear form (Hirota, 1971), the symmetric reduction (Olver, 1986), the Bäcklund transformation (Konno and Wadati, 1975; Lamb, 1974; Wadati and Toda, 1972; Wahlquist and Estabrook, 1973) and the Darboux transformation (Gu *et al.*, 1999; Matveev and Salle, 1991; Rogers and Schief, 2002; Wadati *et al.*, 1975), the formal (Lou and Chen, 1999) and non-formal (Lou, 2000, 2002; Lou and Lu, 1996; Lou and Ruan, 2001; Tang *et al.*, 2002) variable separation methods, etc.

If the soliton structures and properties of the (1 + 1)-dimensional integrable NLPDEs are now understood very well, the study of higher-order dimensional soliton effects is still in its infancy, and continues to attract much attention. For example, the interactions between soliton solutions of (1 + 1)-dimensional integrable models are usually considered to be completely elastic. However, it has been found that for some (2 + 1)-dimensional soliton models, completely non-elastic interactions may occur when specific conditions between the wave vectors and the velocities are satisfied. For instance at a specific time, one soliton may fission to two or more solitons, or on contrarily, two or more solitons may fusion to one soliton. These two types of phenomena are soliton fission and soliton fusion respectively. The same phenomena were also reported in case of (1 + 1)-dimensional models (Wang *et al.*, 2004).

Lin and Qian (2003) investigated the integrable (2 + 1)-dimensional higher-order Broer–Kaup (HBK) system (Lin and Li, 2002). They successfully extended the general variable separation approach (GVSA) to this system of equations and obtained as results that the behaviours of the interactions for the compacton solutions are not completely elastic for some types of them and completely elastic for some others. They also found completely elastic interaction between three solitonic excitations. Li *et al.* (2004) starting from the Bäcklund transformation and using the Cole–Hopf transformation, they reduced the integrable (2 + 1)-dimensional HBK system to a simple linear evolution equation with two arbitrary functions of  $\{x, t\}$  and  $\{y, t\}$ . They found out some special solutions of this system of equations, by choosing the Lou's variable separation approach (VSA). As we know that soliton supplies good applied prospects in many fields of natural science, and the interactive property of soliton plays important role in developing of many applications. Therefore, the study of this interactive property for the integrable models is more significant.

In this paper, based on the Weiss–Tabor–Carnevale (WTC) truncation method and the GVSA, we will present another features of the following integrable (2 + 1)-dimensional HBK system

$$u_t + 4(u_{xx} + u^3 - 3uu_x + 3uw)_{xy} + 12(uv)_{xx} = 0, \quad (1)$$

$$v_t + 4(v_{xx} + 3vu^2 + 3uv_x + 3vw)_x = 0, \quad (2)$$

$$v_x - w_y = 0, \quad (3)$$

which was first obtained from the inner parameter dependent symmetry constraints of the KP equation (Lou and Hu, 1997). When we take  $x = y$ , the system (1)–(3) is reduced to the usual (1 + 1)-dimensional HBK system. Lin (Lin and Li, 2002) has proved that the (2 + 1)-dimensional HBK system possesses the Painlevé Property. Many significant results have obtained in Lin and Qian (2003) and Lin and Li (2002). However, system (1)–(3) possesses many interesting localized structures, which have not yet revealed such as soliton fusion and soliton fission.

The outline of the paper is as follows. In Section 2, a summary of the truncature of Painlevé is given. In Section 3, a general solution including three arbitrary functions is obtained for system (1)–(3) by means of WTC truncation method and GVSA. By choosing appropriately these arbitrary functions, we study interaction properties of solitons. Some conclusion and discussion are given in the last section.

**2. SUMMARY OF THE TRONCATURE OF THE PAINLEVÉ EXPANSION**

It is interesting to see that let the following special solution,

$$v = u_y, \quad w = u_x \tag{4}$$

the system (1)–(3) is decoupled into

$$\{u_t + 4(u_{xx} + u^3 + 3uu_x)_x\}_y = 0. \tag{5}$$

Then following the idea of WTC (Weiss *et al.*, 1983), we begin with the Painlevé expansion of Eq. (5) truncated at the constant term

$$u = \varphi^{-1}u_0 + u_1, \tag{6}$$

where  $\varphi \equiv \varphi(x, y, t)$  is the singular manifold variable,  $u_0$  and  $u_1$  the functions of arguments  $x, y$  and  $t$ . The substitution of (6) into (5) leads to six equations depending of the different powers of  $\varphi$ . At the power  $\varphi^{-5}$  the following results

$$u_0 = \varphi_x, \tag{7}$$

and

$$u_0 = 2\varphi_x, \tag{8}$$

are obtained, while, at the power  $\varphi^0$ , we have

$$\{u_{1t} + 4(u_{1xx} + u_1^3 + 3u_1u_{1x})_x\}_y = 0. \tag{9}$$

By choosing  $u_1 \equiv u_1(x, t)$ , Eq. (9) is satisfied automatically. After that, using (7) into the four remaining equations, the following results are obtained

$$\left\{ \begin{array}{l} \varphi^{-3} : 2\varphi_x \varphi_y \{ \varphi_t + 12(u_1^2 + u_{1x}) \varphi_x + 12u_1 \varphi_{xx} + 4\varphi_{xxx} \} = 0, \\ \varphi^{-2} : \varphi_x \{ \varphi_t + 12(u_1^2 + u_{1x}) \varphi_x + 12u_1 \varphi_{xx} + 4\varphi_{xxx} \}_y \\ \quad + \varphi_y \{ \varphi_t + 12(u_1^2 + u_{1x}) \varphi_x + 12u_1 \varphi_{xx} + 4\varphi_{xxx} \}_x \\ \quad + \varphi_{xy} \{ \varphi_t + 12(u_1^2 + u_{1x}) \varphi_x + 12u_1 \varphi_{xx} + 4\varphi_{xxx} \} = 0, \\ \varphi^{-1} : \{ \varphi_t + 12(u_1^2 + u_{1x}) \varphi_x + 12u_1 \varphi_{xx} + 4\varphi_{xxx} \}_{xy} = 0. \end{array} \right. \tag{10}$$

We have found that the equation obtained at the power  $\varphi^{-4}$  is identically satisfied.

Analysing the system mentioned earlier, we find that all equations in this system are satisfied automatically under the conditions of

$$\varphi_t + 12(u_1^2 + u_{1x}) \varphi_x + 12u_1 \varphi_{xx} + 4\varphi_{xxx} = 0. \tag{11}$$

As for the case  $u_0 = 2\varphi_x$ , no new meaningful results can be obtained through similar analyses.

### 3. GENERAL SOLUTIONS OF THE INTEGRABLE (2 + 1)-DIMENSIONAL HBK SYSTEM

About the linear equation (11) of the system (1)–(3), we can construct many types of special solutions. Because  $u_1(x, t)$  is undetermined function of variable  $x$  and  $t$ , we can select after a careful analysis, an appropriate variable separation hypothesis for the function  $\varphi$  as follows

$$\varphi(x, y, t) = f(x, t)g(y) + h(y), \tag{12}$$

where  $f, g$  and  $h$  are arbitrary functions of indicated variables. Inserting (12) into (11), we have the following variable separated equation

$$f_t + 12(u_1^2 + u_{1x}) f_x + 12u_1 f_{xx} + 4f_{xxx} = 0. \tag{13}$$

Substituting Eq. (12) into Eqs. (4) and (6) along with (7), we obtain a general excitation of the integrable (2 + 1)-dimensional HBK system

$$u = \frac{f_x g}{fg + h} + u_1, \tag{14}$$

$$v = \frac{f_x(g_y h - g h_y)}{(fg + h)^2}, \tag{15}$$

$$w = \frac{g \{ f_{xx}(fg + h) - f_x^2 g \}}{(fg + h)^2} + u_{1x}, \tag{16}$$

with four arbitrary functions  $f(x, t)$ ,  $g(y)$ ,  $h(y)$  and  $u_1(x, t)$ .

Since  $u_1(x, t)$  is arbitrary seed solution and  $f(x, t)$  is arbitrary function, it should be important to check at each step of the resolution that for a selected function of  $f(x, t)$ , that the equation satisfied by  $u_1(x, t)$  by substituting this function in (12), can be really solved.

Because of the arbitrariness of the functions  $f$ ,  $g$  and  $h$  appearing in Eq. (15), the potential  $v$  possesses quite rich coherent structures.

**3.1. Exact Solution Derived from the Trivial Solution  $u_1(x, t) = 0$**

If we take the trivial solution  $u_1(x, t) = 0$  into Eq. (13), then we can solve the obtained equation as

$$f(x, t) = \exp(kx - 4k^3t). \tag{17}$$

By choosing  $g(y) = \exp(l_1y)$  and  $h(y) = \exp(l_2y)$ , the potential field  $v$  can be written as

$$v = \frac{k(l_1 - l_2) \exp(kx - 4k^3t)\exp(l_1 + l_2)y}{\{\exp(kx - 4k^3t)\exp(l_1y) + \exp(l_2y)\}^2}, \tag{18}$$

where  $l_1, l_2$  and  $k$  are arbitrary constants.

**3.2. Exact Solitary Wave Solutions**

Due to the complexity of Eq. (13), we find through our study that there is neither solution of the type of Jacobi elliptic function, nor solution of secant hyperbolic function for  $f$ .

By selecting the following solution for  $f$

$$f(x, t) = a \tanh(kx - \omega t) + b, \tag{19}$$

where  $a, b, k$  and  $\omega$  are constants to be determined later. Inserting (19) into (13) and solving the equation obtained for  $u_1(x, t)$ , leads to

$$u_1(x, t) = k \tanh(kx - \omega t), \tag{20}$$

and

$$u_1(x, t) = 2k \tanh(kx - \omega t), \tag{21}$$

the dispersion relation between  $\omega$  and  $k$  for these two solutions are

$$\omega = 4k^3, \tag{22}$$

$$\omega = 16k^3, \tag{23}$$

respectively, where  $a, b$  and  $k$  are arbitrary constants. Without loss of generality, we will choose  $a = 1, b = 0$ .

Therefore, when the conditions (19)–(23) hold, we can have the following solitary wave solutions for the potential field  $v$ .

Case 1.  $g = \tanh(l_1 y)$ ,  $h = \tanh(l_2 y) + A$ .

$$v = \frac{k \operatorname{sech}^2(kx - \omega t) \{l_1 \operatorname{sech}^2(l_1 y) (\tanh(l_2 y) + A) - l_2 \operatorname{sech}^2(l_2 y) \tanh(l_1 y)\}}{\{\tanh(kx - \omega t) \tanh(l_1 y) + \tanh(l_2 y) + A\}^2} \tag{24}$$

Case 2.  $g = \operatorname{sech}(l_1 y)$ ,  $h = \tanh(l_2 y) + A$ ,

$$v = \frac{k \operatorname{sech}^2(kx - \omega t) \{-l_1 \operatorname{sech}(l_1 y) \tanh(l_1 y) (\tanh(l_2 y) + A) - l_2 \operatorname{sech}^2(l_2 y) \operatorname{sech}(l_1 y)\}}{\{\tanh(kx - \omega t) \operatorname{sech}(l_1 y) + \tanh(l_2 y) + A\}^2}, \tag{25}$$

Case 3.  $g = \tanh(l_1 y)$ ,  $h = \operatorname{sech}(l_2 y) + A$ ,

$$v = \frac{k \operatorname{sech}^2(kx - \omega t) \{l_1 \operatorname{sech}^2(l_1 y) (\operatorname{sech}(l_2 y) + A) - (-l_2 \operatorname{sech}(l_2 y) \tanh(l_2 y)) \tanh(l_1 y)\}}{\{\tanh(kx - \omega t) \tanh(l_1 y) + \operatorname{sech}(l_2 y) + A\}^2}, \tag{26}$$

Case 4.  $g = \operatorname{sech}(l_1 y)$ ,  $h = \operatorname{sech}(l_2 y) + A$ ,

$$v = \frac{k \operatorname{sech}^2(kx - \omega t) \{-l_1 \operatorname{sech}(l_1 y) \tanh(l_1 y) (\operatorname{sech}(l_2 y) + A) - (-l_2 \operatorname{sech}(l_2 y) \tanh(l_2 y)) \operatorname{sech}(l_1 y)\}}{\{\tanh(kx - \omega t) \operatorname{sech}(l_1 y) + \operatorname{sech}(l_2 y) + A\}^2}, \tag{27}$$

where  $A$ ,  $l_1$  and  $l_2$  are arbitrary constants.

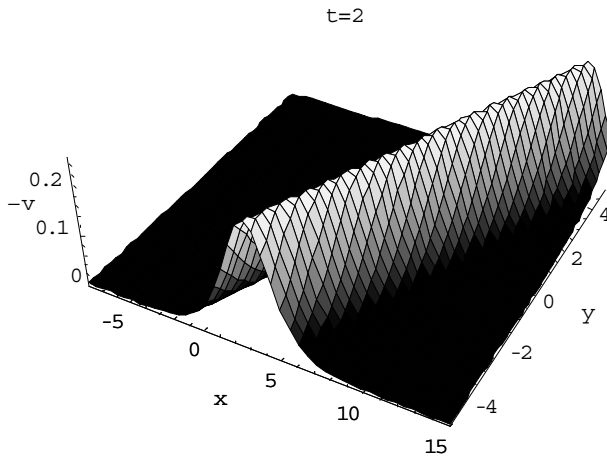


Fig. 1. The graph of the potential field  $v$  for the coherent structure (18) obtained for the given values,  $k = 1$ ,  $l_1 = 1$ ,  $l_2 = 2$ .

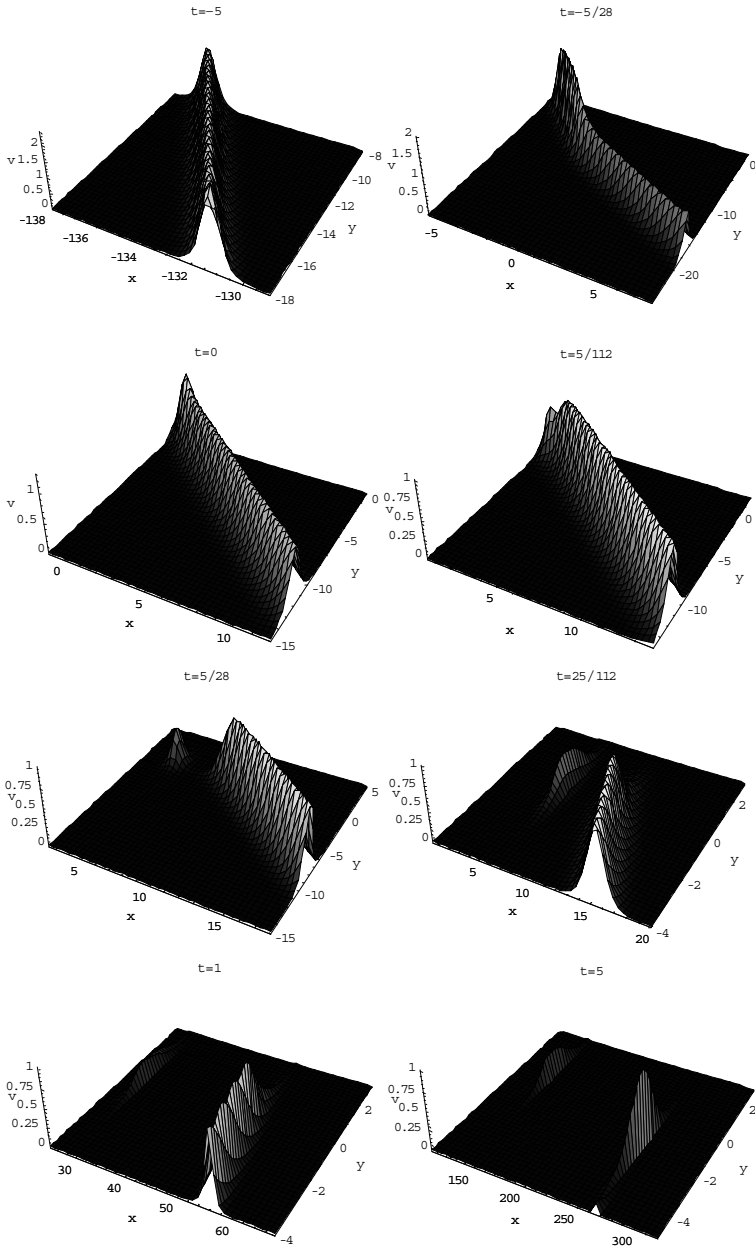
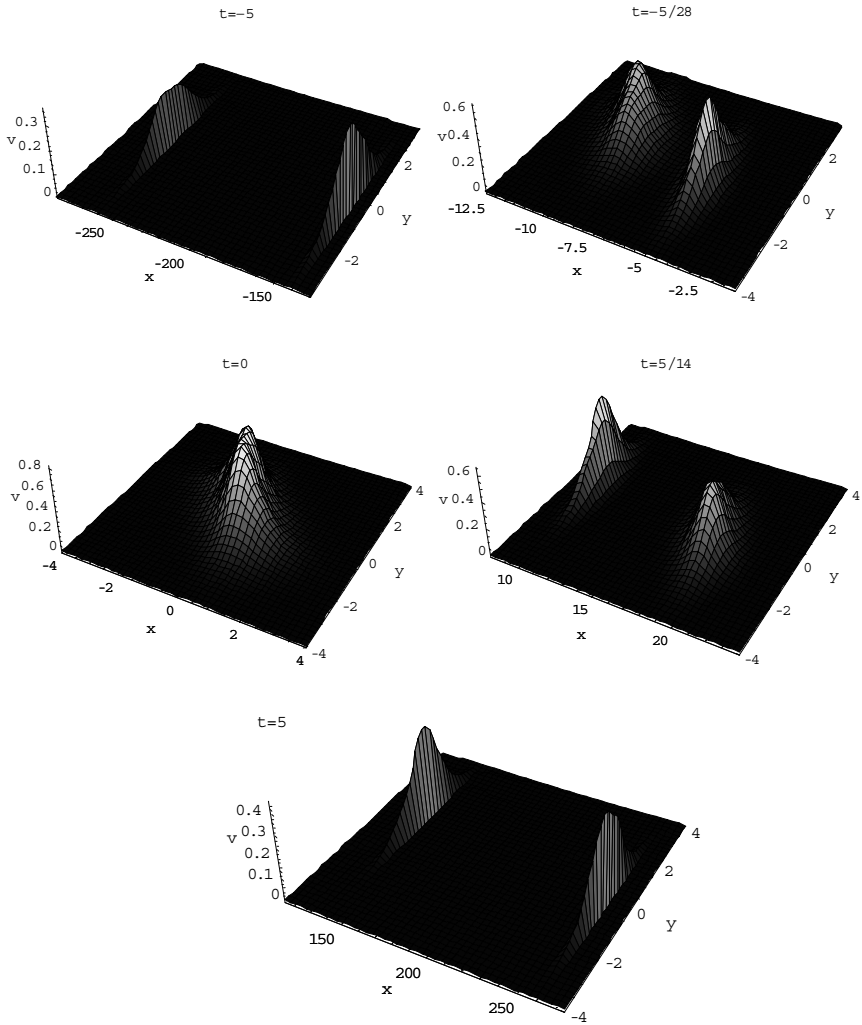


Fig. 2. The steps of the fission of the solitonic excitation for the potential field  $v$  given by solution (31).



**Fig. 3.** Completely elastic interaction between two dromion solutions for the potential field  $v$  expressed by (32).

### 3.3. Exact Solutions Leading to Interaction of Waves

As for the interaction of the wave solutions, a hypothesis is adopted. Namely, just suppose

$$f(x, t) = a \tanh(k_1x - \omega_1t) + b \tanh(k_2x - \omega_2t) + c, \quad (28)$$



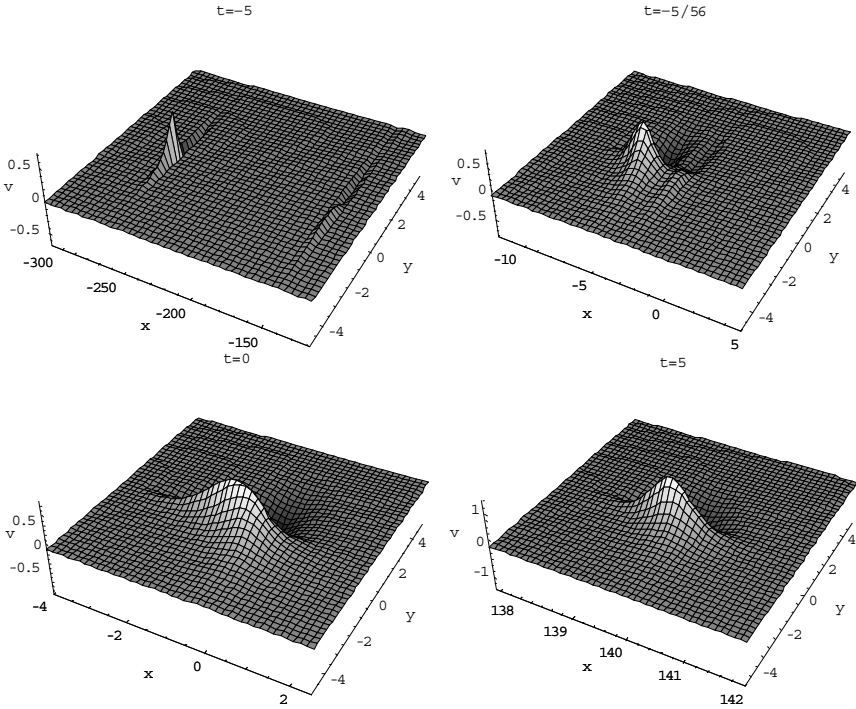


Fig. 4. Two-solitonic excitations fusion for the potential field  $v$  given by (33).

where  $a, b, c, k_1, k_2, \omega_1$  and  $\omega_2$  are constants to be determined later. Inserting (28) into (13) and solving the equation obtained for  $u_1(x, t)$ , leads to

$$u_1(x, t) = k_1 \tanh(k_1x - \omega_1t) + k_2 \tanh(k_2x - \omega_2t), \tag{29}$$

with the dispersion relation

$$\omega_1 = 4k_1(k_1^2 + 3k_2^2), \quad \omega_2 = 4k_2(k_2^2 + 3k_1^2), \tag{30}$$

where  $a, b, c, k_1$  and  $k_2$  are arbitrary constants. In what follows, we may choose without loss of generality  $a = 1, b = 2$  and  $c = 0$

When the conditions (28)–(30) are satisfied, the potential  $v$  can be obtained by the following selection of the functions  $g(y)$  and  $h(y)$ .

Case 1.  $g = \tanh(l_1y), h = \tanh(l_2y) + A$ . From Eq. (15), we have

$$v = \frac{(k_1 \operatorname{sech}^2(k_1x - \omega_1t) + 2k_2 \operatorname{sech}^2(k_2x - \omega_2t)) \times \{l_1 \operatorname{sech}^2(l_1y)(\tanh(l_2y) + A) - l_2 \operatorname{sech}^2(l_2y) \tanh(l_1y)\}}{\{(\tanh(k_1x - \omega_1t) + 2 \tanh(k_2x - \omega_2t)) \tanh(l_1y) + \tanh(l_2y) + A\}^2}. \tag{31}$$

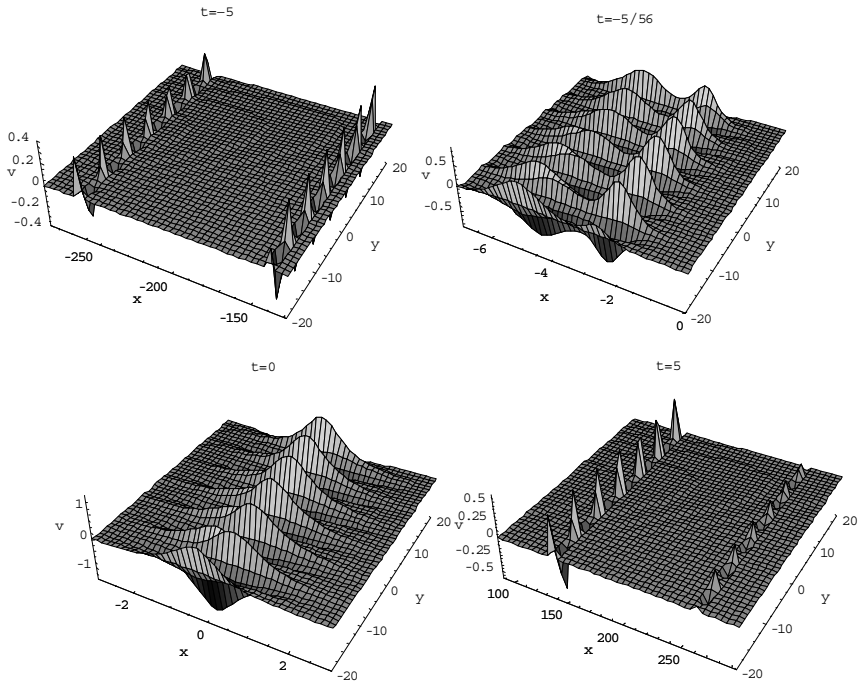


Fig. 5. The evolution of  $y$ -periodic solitons (35).

Case 2.  $g = \operatorname{sech}(l_1 y)$ ,  $h = \tanh(l_2 y) + A$ . Other new soliton excitation solution of Eq. (15) is given by

$$v = \frac{(k_1 \operatorname{sech}^2(k_1 x - \omega_1 t) + 2k_2 \operatorname{sech}^2(k_2 x - \omega_2 t)) \times \{-l_1 \operatorname{sech}(l_1 y) \tanh(l_1 y) (\tanh(l_2 y) + A) - l_2 \operatorname{sech}^2(l_2 y) \operatorname{sech}(l_1 y)\}}{\{(\tanh(k_1 x - \omega_1 t) + 2 \tanh(k_2 x - \omega_2 t)) \operatorname{sech}(l_1 y) + \tanh(l_2 y) + A\}^2}. \tag{32}$$

Case 3.  $g = \tanh(l_1 y)$ ,  $h = \operatorname{sech}(l_2 y) + A$ . From Eq. (15), we have

$$v = \frac{(k_1 \operatorname{sech}^2(k_1 x - \omega_1 t) + 2k_2 \operatorname{sech}^2(k_2 x - \omega_2 t)) \times \{l_1 \operatorname{sech}^2(l_1 y) (\operatorname{sech}(l_2 y) + A) - (-l_2 \operatorname{sech}(l_2 y) \tanh(l_2 y)) \tanh(l_1 y)\}}{\{(\tanh(k_1 x - \omega_1 t) + 2 \tanh(k_2 x - \omega_2 t)) \tanh(l_1 y) + \operatorname{sech}(l_2 y) + A\}^2}. \tag{33}$$

Case 4.  $g = \operatorname{sech}(l_1 y)$ ,  $h = \operatorname{sech}(l_2 y) + A$ . Other new type of solution for the potential (15) is

$$v = \frac{(k_1 \operatorname{sech}^2(k_1 x - \omega_1 t) + 2k_2 \operatorname{sech}^2(k_2 x - \omega_2 t)) \{-l_1 \operatorname{sech}(l_1 y) \times \tanh(l_1 y) (\operatorname{sech}(l_2 y) + A) - (-l_2 \operatorname{sech}(l_2 y) \tanh(l_2 y)) \operatorname{sech}(l_1 y)\}}{\{(\tanh(k_1 x - \omega_1 t) + 2 \tanh(k_2 x - \omega_2 t)) \operatorname{sech}(l_1 y) + \operatorname{sech}(l_2 y) + A\}^2}. \tag{34}$$

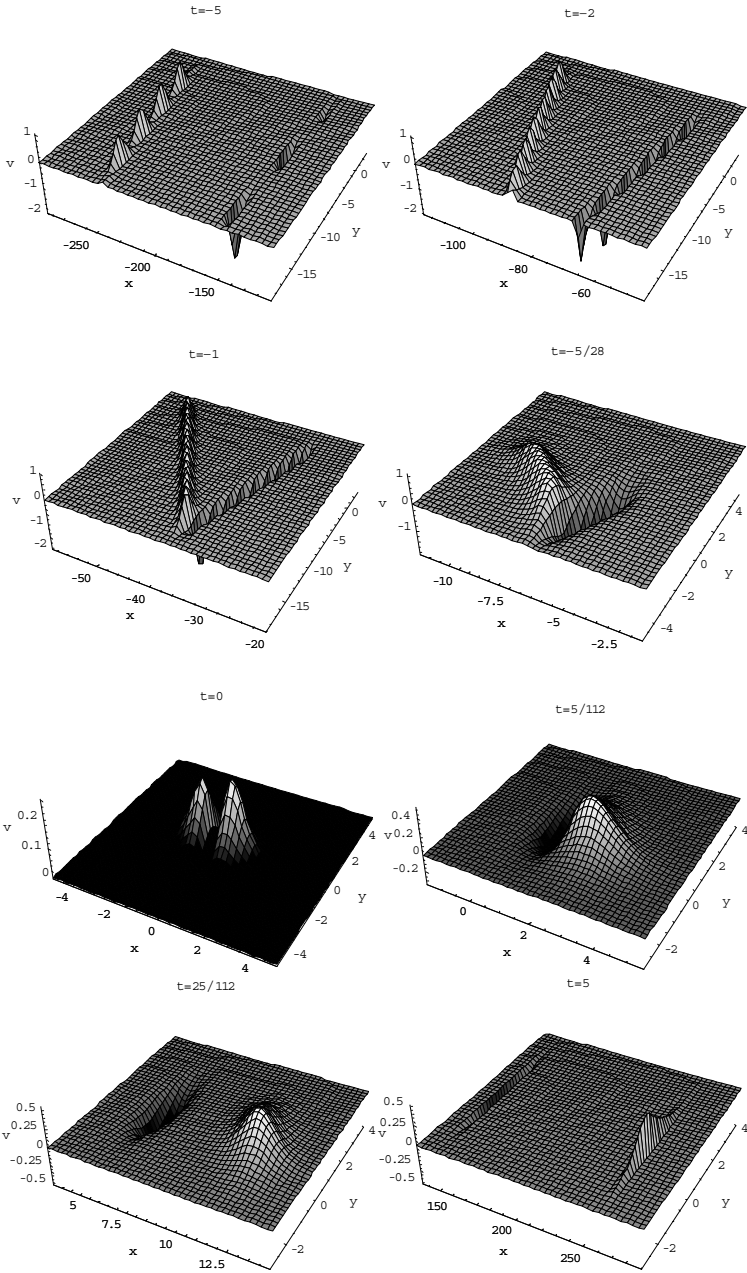
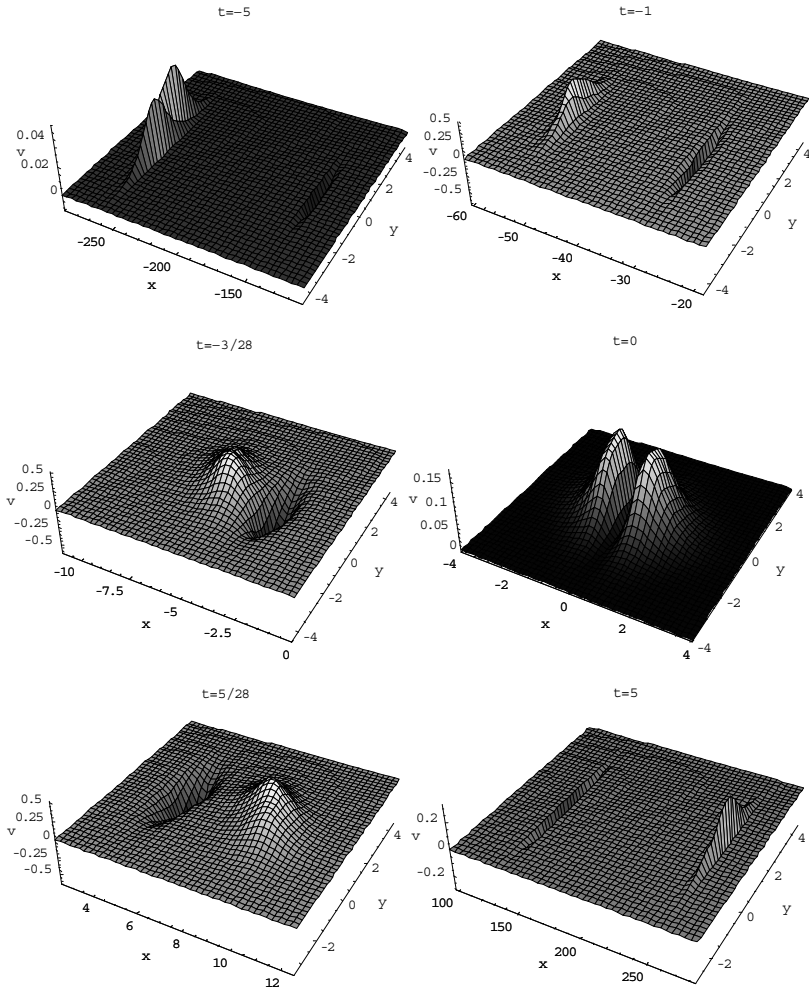


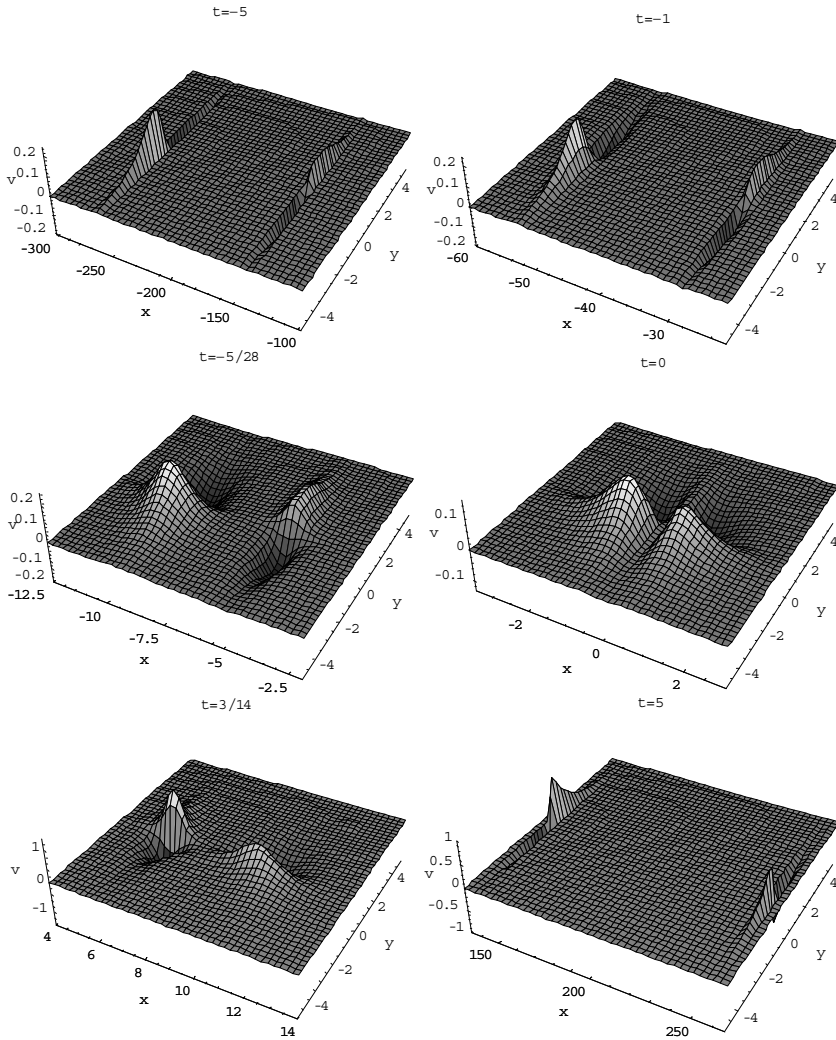
Fig. 6. Steps of the fusion of solitonic excitations for the potential field  $v$  expressed by (31).



**Fig. 7.** The evolution of the dromion excitations of potential field  $v$  expressed by (32).

Other class of solutions can be obtained under the following choice of the functions  $g(y)$  and  $h(y)$ .

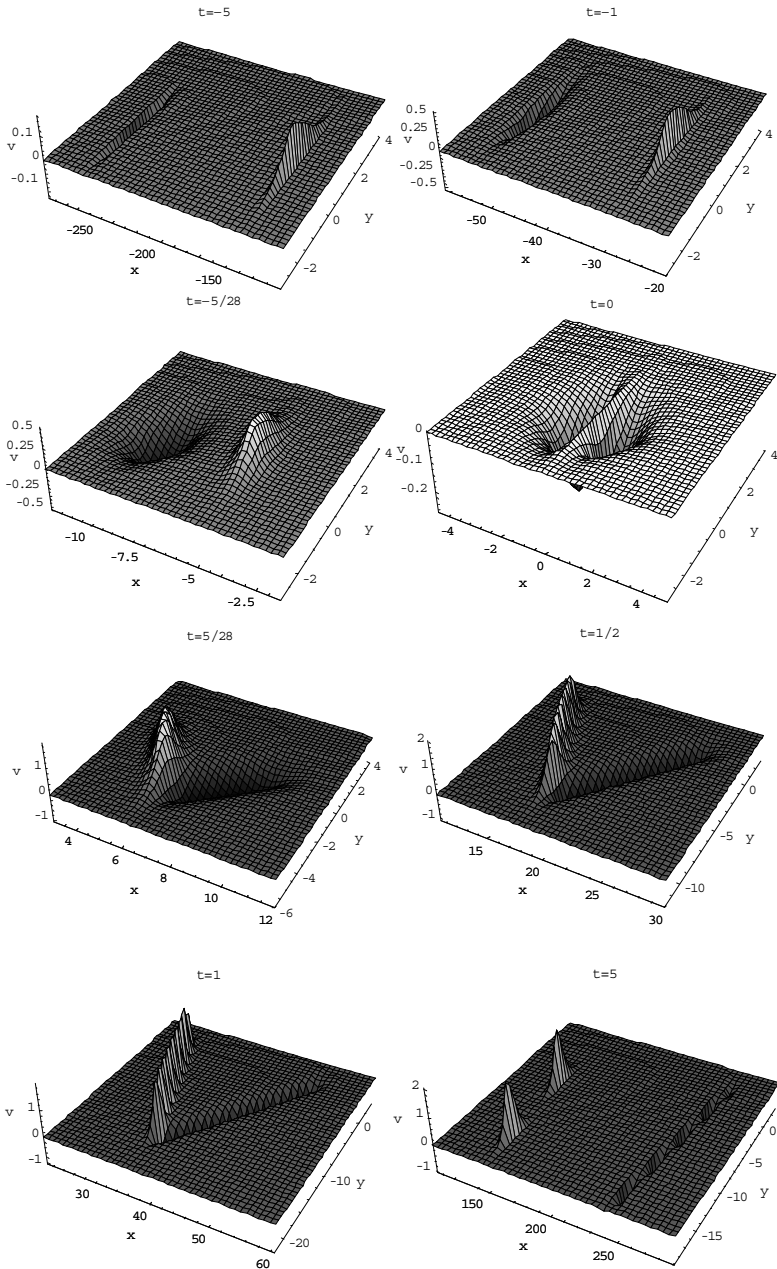
- Case 5.  $g = \cos(l_1 y), h = \cos(l_2 y) + A,$
- Case 6.  $g = \operatorname{sech}(l_1 y), h = \cos(l_2 y) + A,$
- Case 7.  $g = \cos(l_1 y), h = \operatorname{sech}(l_2 y) + A,$
- Case 8.  $g = \tanh(l_1 y), h = \cos(l_2 y) + A,$
- Case 9.  $g = \cos(l_1 y), h = \tanh(l_2 y) + A.$



**Fig. 8.** The evolution of the two-solitonic excitations of potential field  $v$  given by (33).

If we take for example case 5, we can write down explicitly the following form of solution for the potential  $v$

$$v = \frac{(k_1 \operatorname{sech}^2(k_1x - \omega_1t) + 2k_2 \operatorname{sech}^2(k_2x - \omega_2t)) \times \{-l_1 \sin(l_1y)(\cos(l_2y) + A) - (-l_2 \sin(l_2y)) \cos(l_1y)\}}{\{(\tanh(k_1x - \omega_1t) + 2 \tanh(k_2x - \omega_2t)) \cos(l_1y) + \cos(l_2y) + A\}^2}. \quad (35)$$



**Fig. 9.** Steps of the fission of solitonic excitations of potential field  $v$  expressed by (31).

From Fig. 1 we see that it approaches to zero at all directions except two special directions. This behaviour is quite close to that of the solitoff which usually approaches to zero at all directions except a special direction.

Figures 2–5 show the detail on the interaction property of the solitonic excitations obtained for the potential field  $v$  (15), when selecting the parameter values  $k_1 = 2, l_1 = 1, k_2 = 1, l_2 = 2$  and  $A = 4$ , and the corresponding time is indicated on these figures.

Figures 6–8 are obtained for the parameters values  $k_1 = -2, l_1 = 1, k_2 = 1, l_2 = 2$  and  $A = 4$ , and the corresponding time is indicated on these figures.

Figure 9 is obtained for the selected values  $k_1 = 2, l_1 = 1, k_2 = -1, l_2 = 2$  and  $A = 4$ .

#### 4. CONCLUSION

We have sought analytically solutions for the integrable  $(2 + 1)$ -dimensional HBK system by means of the WTC truncation and the general variable separation approach. By selecting the three arbitrary functions which appear in the solution, we have studied numerically the interaction properties of the localized solutions (dromions, solitons, periodic solitons) moving with different velocities. We have found some interactions completely elastic, i.e. the solitonic excitations pass through each other and preserve their shapes and velocities, the only change being the phase shift. We have also found during the interaction; on the one hand that several solitons have fused to one, on the other hand, on the contrary, one single soliton has fission into several solitons. These fusion and fission phenomena have not yet reported before on the integrable  $(2 + 1)$ -dimensional HBK system.

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